

DIFFERENTIAL INEQUALITIES ON HOMOGENEOUS TORIC BUNDLES

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ABSTRACT. We study a generalized Abreu Equation in n -dimensional polytopes and prove some differential inequalities for homogeneous toric bundles.

Keywords. generalized Abreu Equation, differential inequalities.

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1. INTRODUCTION

The existence of extremal and constant scalar curvature is a central problem in Kähler geometry. In a series of papers [11], [12], [13], and [14], Donaldson studied this problem on toric manifolds and proved the existence of metrics of constant scalar curvatures on toric surfaces under an appropriate stability condition. Later on in [7] and [8], Chen, Li and Sheng proved the existence of metrics of prescribed scalar curvatures on toric surfaces under the uniform stability condition.

It is important to generalize the results of Chen, Li and Sheng to more general Kähler manifold. This is one of a sequence of papers, aiming at generalizing the results of Chen, Li and Sheng to homogeneous toric bundles. A homogeneous toric bundle $G \times_K M$ have a compact toric manifold M as fiber and a generalized flag manifold G/K as basis. Let $U \subset M$ be any complex coordinate domain and let $G(U) \subset G \times_K M$ be the image of U under the left G -action on $G \times_K M$. In this paper we establish some differential inequalities on $G(U)$, which are generalizations of the differential inequalities in [7]. The differential inequalities of this paper play an important role in our works on scalar curvatures for homogeneous toric bundles.

2. HOMOGENEOUS TORIC BUNDLES

We recall some facts about homogeneous toric bundles and refer to [20], [4] and [15] for details. Let G be a compact semisimple Lie group, K be the centralizer of a torus S in G . Let T be a maximal torus in G containing S . Then $T \subset C(S) = K$, G/K is a generalized flag manifold. Denote $\mathfrak{o} = \mathfrak{e}K$. Let \mathfrak{g} (resp. \mathfrak{k} , \mathfrak{h}) be the Lie algebra of G

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(resp. K, T). Let B denote the Killing form of \mathfrak{g} . Recall that $-B$ is a positive definite inner product on \mathfrak{g} . There is a orthogonal decomposition with respect to $-B$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad \text{Ad}(k)\mathfrak{m} \subset \mathfrak{m} \text{ for all } k \in K.$$

The tangent space of G/K at \mathfrak{o} is identified with \mathfrak{m} .

Let $Z(K)$ be the center of K , it is an n -dimensional torus, denoted by T^n . Let (M, ω) be a compact toric Kähler manifold of complex dimension n , where T^n acts effectively on M . Let $\varrho : K \rightarrow T^n$ be a surjective homomorphism. The homogeneous toric bundle $G \times_K M$ is defined to be the space $G \times M$ modulo the relation $(gh, x) = (g, \varrho(h)x)$, where $g \in G, h \in K$, and $x \in M$. Later we will omit ϱ to simplify notations. The space $G \times_K M$ is a fiber bundle with fiber M and base space G/K , a generalized flag manifold. There is a natural G -action on $G \times_K M$ given by $g \cdot [h, x] = [gh, x]$, $g \in G, x \in M$, and a natural T^n -action on $G \times_K M$ given by $k \cdot [g, x] = [g, k^{-1}x]$, $k \in T^n$. Both G/K and $G \times_K M$ are complex manifolds. In fact, they can be expressed as $G^\mathbb{C}/P$ and $G^\mathbb{C} \times_P M$ respectively, where $G^\mathbb{C}$ is the complexification of G , P is a parabolic subgroup of $G^\mathbb{C}$.

Denote by $\mathfrak{g}^\mathbb{C}, \mathfrak{k}^\mathbb{C}, \mathfrak{h}^\mathbb{C}$ the complexification of $\mathfrak{g}, \mathfrak{k}, \mathfrak{h}$ respectively. Let R be the root system of $\mathfrak{g}^\mathbb{C}$ with respect to $\mathfrak{h}^\mathbb{C}$, we have the root space decomposition

$$\mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \sum_{\alpha \in R} \mathbb{C}E_\alpha = \mathfrak{k}^\mathbb{C} \oplus \mathfrak{m}^\mathbb{C},$$

$$\mathfrak{k}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \sum_{\alpha \in R_K} \mathbb{C}E_\alpha, \quad \mathfrak{m}^\mathbb{C} = \sum_{\alpha \in R_M} \mathbb{C}E_\alpha,$$

where R_K is a subset of R and $R_M = R \setminus R_K$. For any $\varphi \in \mathfrak{h}^*$ we define $h_\varphi \in \mathfrak{h}$ by

$$B(h, h_\varphi) = \varphi(h) \text{ for all } h \in \mathfrak{h},$$

and set

$$H_\varphi = \sqrt{-1}h_\varphi.$$

Define

$$\mathfrak{t} = \mathfrak{h} \cap \mathfrak{z}(\mathfrak{k}^\mathbb{C}),$$

Let

$$\kappa : \mathfrak{h}^* \rightarrow \mathfrak{t}^* \quad \alpha \mapsto \alpha|_{\mathfrak{t}}$$

be the restriction map. Set $R_T = \kappa(R) = \kappa(R_M)$. The elements of R_T are called T -roots.

Choose a fundamental system $\Pi = \{\alpha_1, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+h}\}$ of R . We fix a lexicographic ordering and let R^+ (resp. R_K^+) be the set of positive roots of R (resp.

R_K) with respect to Π . Let $R_M^+ = R^+ \setminus R_K^+$. Denote $\Pi_K = \Pi \cap R_K$. We assume that $\Pi_K = \{\alpha_{n+1}, \dots, \alpha_{n+\hbar}\}$. For $i \leq n$, let

$$\tilde{h}_{\alpha_i} = h_{\alpha_i} + \sum_{j=n+1}^{n+\hbar} d_i^j h_{\alpha_j},$$

where d_i^j are constants such that $B(\tilde{h}_{\alpha_i}, h_{\alpha_j}) = 0$ for all $j = n+1, \dots, n+\hbar$. Denote

$$H_{\alpha_i} = \sqrt{-1}h_{\alpha_i}, \quad i = n+1, \dots, n+\hbar, \quad \tilde{H}_{\alpha_j} = \sqrt{-1}\tilde{h}_{\alpha_j}, \quad j = 1, \dots, n.$$

For any $\alpha \in R_K$, we have $\alpha(\tilde{H}_{\alpha_j}) = 0$. Then $\mathfrak{t} = \text{span}\{\tilde{H}_{\alpha_1}, \dots, \tilde{H}_{\alpha_n}\}$. Let $\mathfrak{h}' = \text{span}\{H_{\alpha_{n+1}}, \dots, H_{\alpha_{n+\hbar}}\}$. We have the orthogonal decomposition with respect to $-B$: $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{h}'$.

We choose a Weyl basis $e_\alpha \in \mathfrak{g}_\alpha^\mathbb{C}$ of $\mathfrak{g}^\mathbb{C}$ such that, for $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$,

$$(2.1) \quad B(e_\alpha, e_{-\alpha}) = 1, \quad [e_\alpha, e_{-\alpha}] = h_\alpha,$$

Obviously, $[h, e_\alpha] = \alpha(h)e_\alpha$, for any $h \in \mathfrak{t}$. Set

$$V_\alpha = e_\alpha - e_{-\alpha}, \quad W_\alpha = \sqrt{-1}(e_\alpha + e_{-\alpha}), \quad H_\alpha = \sqrt{-1}h_\alpha.$$

It is easy to see that $H_\alpha, V_\alpha, W_\alpha \in (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) \cap \mathfrak{g}$. For any $\alpha \in R^+$ we have $\alpha = \sum_{j=1}^{n+\hbar} M_\alpha^j \alpha_j$ with $M_\alpha^j \geq 0$. It is easy to see that

$$(2.2) \quad H_\alpha = \sum_{j=1}^{n+\hbar} M_\alpha^j H_{\alpha_j} = \sum_{j=1}^n M_\alpha^j \tilde{H}_{\alpha_j} + \sum_{j=n+1}^{n+\hbar} M_\alpha'^j H_{\alpha_j},$$

where $M_\alpha'^j = M_\alpha^j - \sum_{i=1}^n M_\alpha^i d_i^j$. Obviously, for any $\alpha \in R_{M^+}$, we have

$$(2.3) \quad \sum_{j=1}^n M_\alpha^j > 0.$$

3. (G, T^n) -INVARIANT KÄHLER METRICS

From now on, our convention for the ranges of indices is the following:

$$1 \leq A, B, \dots \leq n+l$$

$$1 \leq i, j, k, \dots \leq n,$$

$$n+1 \leq \alpha \leq n+l$$

where l is the dimension of \mathfrak{m} . For any $1 \leq j \leq n, \alpha \in R_M^+$, let $\tilde{H}_{\alpha_j}^*, V_\alpha^*, W_\alpha^*$ be the fundamental vector fields corresponding to $\tilde{H}_{\alpha_j}, V_\alpha, W_\alpha$. Then the left-invariant vector fields $\{\frac{\partial}{\partial x^j}, \tilde{H}_{\alpha_j}^*, V_\alpha^*, W_\alpha^*\}_{1 \leq j \leq n, \alpha \in R_M^+}$ is a local base of $G \times_K M$. Obviously,

$$(3.1) \quad [\frac{\partial}{\partial x^j}, \tilde{H}_j^*] = 0, \quad [\frac{\partial}{\partial x^j}, V_\alpha^*] = [\frac{\partial}{\partial x^j}, W_\alpha^*] = 0, \quad 1 \leq j \leq n, \alpha \in R_{M^+}.$$

Let $\{dx^j, \nu^j, dV^\alpha, dW^\alpha\}_{1 \leq j \leq n, \alpha \in R_M^+}$ be the dual left-invariant 1-form of the base. Set

$$S_j = \frac{1}{2}(\frac{\partial}{\partial x^j} - \sqrt{-1}H_j^*), \quad 1 \leq j \leq n$$

$$S_\alpha = \frac{1}{2}(V_\alpha^* - \sqrt{-1}W_\alpha^*), \quad \alpha \in R_{M^+}.$$

We define the almost complex structure J by,

$$JS_i = \sqrt{-1}S_i, \quad JS_\alpha = \sqrt{-1}S_\alpha, \quad \forall 1 \leq i \leq n, \alpha \in R_{M^+}.$$

Then J is a (G, T^n) -invariant complex structure. It is easy to see that S_j, S_α are $(1, 0)$ -vector fields. Denote ω^j, ω^α the dual $(1, 0)$ -form of S_j, S_α . It is easy to see that S_α, \bar{S}_α is induced vector field of e_α and $-e_{-\alpha}$. By (2.1) we have

$$(3.2) \quad [S_j, \bar{S}_k] = [S_j, S_k] = 0, \quad [S_\alpha, \bar{S}_\alpha] = -\sqrt{-1}H_\alpha^* = \sum_j M_\alpha^j (S_j - \bar{S}_j),$$

$$(3.3) \quad [S_j, S_\alpha] = -\frac{1}{2}\alpha(\tilde{h}_j)S_\alpha, \quad [S_j, \bar{S}_\alpha] = \frac{1}{2}\alpha(\tilde{h}_j)\bar{S}_\alpha, \quad [S_\alpha, S_\beta] = -N_{\alpha\beta}S_{\alpha+\beta},$$

$$(3.4) \quad [S_\alpha, \bar{S}_\beta] = N_{\alpha-\beta}S_{\alpha-\beta}, \quad \alpha > \beta \quad [S_\alpha, \bar{S}_\beta] = -N_{\alpha-\beta}\bar{S}_{\beta-\alpha}, \quad \alpha < \beta.$$

As in [19] one can check that

$$(3.5) \quad d\nu^j = \sum_{\alpha \in R_{M^+}} 2M_\alpha^j dV^\alpha \wedge dW^\alpha = \sum_{\alpha \in R_{M^+}} M_\alpha^j \sqrt{-1}\omega^\alpha \wedge \bar{\omega}^\alpha.$$

In fact, we have

$$d\nu^j(S_A, \bar{S}_B) = S_A(\nu^j(\bar{S}_B)) - \bar{S}_B(\nu^j(S_A)) - \nu^j([S_A, \bar{S}_B]) = -\nu^j([S_A, \bar{S}_B]).$$

Then (3.5) follows from (3.2)-(3.4).

Denote by $\tau : M \rightarrow \bar{\Delta} \subset \mathfrak{t}^*$ the moment map of M , where Δ is a Delzant polytope. The left invariant 1-form $\{\nu^1, \dots, \nu^n\}$ can be seen as a basis of \mathfrak{t}^* . The moment map $\tau : M \rightarrow \mathfrak{t}^*$ has components, relative to this basis of \mathfrak{t}^* , which we denote by τ_i . Note that $\sum_{i=1}^n \tau_i \nu^i$ is independent of the choice of the bases.

Now we fit G into this picture. We fix a point $o \in \mathbb{R}^n$ and identify \mathfrak{t}^* with $T_o \mathbb{R}^n$. We choose $\{o, \nu^i, i = 1, \dots, n\}$ as a base of \mathbb{R}^n . Let $\xi = (\xi_1, \dots, \xi_n)$ be the coordinate system

with respect to the bases. We choose $\bar{\Delta} \subset \{(\xi_1, \dots, \xi_n) | \xi_1 > 0, \xi_2 > 0, \dots, \xi_n > 0\}$ such that

$$(3.6) \quad \sum_{\alpha \in R_{M+}} \frac{\sum_{j=1}^n M_{\alpha}^j \text{diam}(\Delta)}{D_{\alpha}} < \frac{n}{4},$$

where

$$D_{\alpha} := 2 \sum_{i=1}^n \tau_i \nu^i = 2 \sum_{j=1}^n M_{\alpha}^j \xi_j > 0 \quad \forall \xi \in \bar{\Delta}.$$

Since the moment map is equivariant we can also regard τ as a map from $G \times_K M$ to \mathfrak{t}^* and the components τ_i as functions on $G \times_K M$, that is, we extend $\tau : G \times_K M \rightarrow \bar{\Delta}$ by $\tau([g, x]) = \tau(x)$. Following Donaldson ([14]) we consider the following form

$$\Omega = d \left(\sum_{i=1}^n \tau_i \nu^i \right)$$

in $\tau^{-1}(\Delta)$. By (3.5) we can write it as

$$(3.7) \quad \Omega = \sum_{i=1}^n d\tau_i \wedge \nu^i + \sum_{\alpha \in R_{M+}} D_{\alpha} (dV^{\alpha} \wedge dW^{\alpha}).$$

It is easy to see that J is Ω -compatible. i.e.,

$$\Omega(X, JX) > 0, \quad \forall X \neq 0, \quad \Omega(JX, JY) = \Omega(X, Y).$$

So $(G \times_K M, \Omega)$ is a Kähler manifold with the Kähler form Ω . It is well known that there exists a convex function f such that $\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial \tau_i}{\partial x^j}$. Then the Kähler form Ω can be written as

$$(3.8) \quad \begin{aligned} \Omega &= \sum_{i,j=1}^n f_{ij} dx^i \wedge \nu^j + \sum_{\alpha \in R_{M+}} D_{\alpha} (dV^{\alpha} \wedge dW^{\alpha}) \\ &= \frac{1}{2} \left[\sum_{i,j=1}^n f_{ij} \omega^i \wedge \bar{\omega}^j + \sum_{\alpha \in R_{M+}} D_{\alpha} (\omega^{\alpha} \wedge \bar{\omega}^{\alpha}) \right]. \end{aligned}$$

We denote Ω by Ω_f . Then the Riemannian metric is given by

$$\begin{aligned} \mathcal{G}_f &= \sum_{i,j=1}^n f_{ij} (dx^i \otimes dx^j + \nu^i \otimes \nu^j) + \sum_{\alpha \in R_{M+}} D_{\alpha} (dV^{\alpha} \otimes dV^{\alpha} + dW^{\alpha} \otimes dW^{\alpha}) \\ &= \sum_{i,j} \frac{f_{ij}}{2} (\omega^i \otimes \bar{\omega}^j + \bar{\omega}^j \otimes \omega^i) + \sum_{\alpha \in R_{M+}} \frac{D_{\alpha}}{2} (\omega^{\alpha} \otimes \bar{\omega}^{\alpha} + \bar{\omega}^{\alpha} \otimes \omega^{\alpha}). \end{aligned}$$

Let

$$\xi_i = \frac{\partial f}{\partial x^i}, \quad u(\xi_1, \dots, \xi_n) = \sum_{i=1}^n x^i \xi_i - f(x).$$

Then

$$\mathcal{G}_u = \sum_{i,j=1}^n (u_{ij} d\xi_i \otimes d\xi_j + u^{ij} \nu^i \otimes \nu^j) + \sum_{\alpha \in R_{M+}} D_\alpha (dV^\alpha \otimes dV^\alpha + dW^\alpha \otimes dW^\alpha),$$

where $u_{ij} = \frac{\partial^2 u}{\partial \xi_i \partial \xi_j}$ and (u^{ij}) is the inverse matrix of (u_{ij}) . The Kähler form Ω and the Kähler metric \mathcal{G} can be extended over $G \times_K M$.

Suppose that Δ is defined by linear inequalities $h_k(\xi) - c_k > 0$, for $k = 1, \dots, d$, where c_k are constants and h_k are affine linear functions in \mathbb{R}^n , $k = 1, \dots, d$, and each $h_k(\xi) - c_k = 0$ defines a facet of Δ . Write $\delta_k(\xi) = h_k(\xi) - c_k$ and set

$$(3.9) \quad v(\xi) = \sum_k \delta_k(\xi) \log \delta_k(\xi).$$

It defines a Kähler metric on $G \times_K M$, which we call the Guillemin metric.

4. GENERALIZED ABREU EQUATIONS

In this section we calculate the Ricci tensor and the scalar curvature of $G \times_K M$. Podesta and Spiro have calculated the Ricci tensor in [20]. Here we use a different method. First we note that for any $(1, 0)$ -vector fields X, Y , we have $\nabla_X Y$ is also a $(1, 0)$ -vector field. In fact, let (z^1, \dots, z^{n+l}) be holomorphic coordinates. Then $X = \sum_{i=1}^{n+l} X^i \frac{\partial}{\partial z^i}$ and $Y = \sum_{i=1}^{n+l} Y^i \frac{\partial}{\partial z^i}$. Here X^i, Y^i may not be holomorphic functions. Then

$$\nabla_{\bar{X}} Y = \sum_{i,j} \bar{X}^i \left(Y^j \nabla_{\frac{\partial}{\partial \bar{z}^i}} \frac{\partial}{\partial z^j} + \frac{\partial Y^j}{\partial \bar{z}^i} \frac{\partial}{\partial z^j} \right) = \sum_{i,j} \bar{X}^i \frac{\partial Y^j}{\partial \bar{z}^i} \frac{\partial}{\partial z^j}.$$

We want find a smooth holomorphic $(n, 0)$ -field $L (\wedge_{j=1}^n S_j) \wedge (\wedge_{\alpha \in R_{M+}} S_\alpha)$ on $\tau^{-1}(\Delta)$, where L is a smooth function on $\tau^{-1}(\Delta)$. By the Koszul formula, for any $j \leq n$, $\alpha \in R_{M+}$, we have

$$\begin{aligned} (4.1) \quad & g(\nabla_{\bar{S}_j} S_\alpha, \bar{S}_\alpha) \\ &= \frac{1}{2} [\bar{S}_j g(S_\alpha, \bar{S}_\alpha) - g(\bar{S}_j, [S_\alpha, \bar{S}_\alpha]) + g(S_\alpha, [\bar{S}_\alpha, \bar{S}_j]) + g(\bar{S}_\alpha, [\bar{S}_j, S_\alpha])] \\ &= \frac{1}{2} \left[\frac{1}{4} \frac{\partial D_\alpha}{\partial x^j} - g(\bar{S}_j, -\sqrt{-1} \sum_{k=1}^n M_\alpha^k H_k) \right] + \frac{1}{4} D_\alpha \alpha(\tilde{h}_j) = \frac{1}{4} D_\alpha \alpha(\tilde{h}_j). \end{aligned}$$

Similarly we have for any $\alpha, \beta \in R_{M+}$, $j, k, l \leq n$,

$$(4.2) \quad g(\nabla_{\bar{S}_j} S_l, \bar{S}_k) = g(\nabla_{\bar{S}_\alpha} S_l, \bar{S}_k) = g(\nabla_{\bar{S}_\alpha} S_\beta, \bar{S}_\beta) = 0.$$

Denote $\mathbb{T} = (\wedge_{j=1}^n S_j) \wedge (\wedge_{\alpha \in R_{M^+}} S_\alpha)$. By (4.1) and (4.2), it is easy to see that

$$\nabla_{\bar{S}_\beta} \mathbb{T} = 0, \quad \nabla_{\bar{S}_j} \mathbb{T} = \left(\sum_{\alpha \in R_{M^+}} \frac{\alpha(\tilde{h}_j)}{2} \right) \mathbb{T}.$$

Denote $\sigma_i = 2 \sum_{\alpha \in R_{M^+}} \alpha(\tilde{h}_i)$. Choose $L = e^{-\frac{\sum_{i=1}^n \sigma_i x^i}{2}}$, then

$$\nabla_{\bar{S}_\beta}(L\mathbb{T}) = 0, \quad \nabla_{\bar{S}_i}(L\mathbb{T}) = 0, \quad i \leq n, \alpha \in R_{M^+}.$$

It follows that

$$e^{\sum_{i=1}^n \sigma_i x^i} (\wedge_{j=1}^n \omega^j) \wedge (\wedge^{\alpha \in R_{M^+}} \omega_\alpha) = |h|^2 (dz^i \wedge d\bar{z}^i)^{n+l},$$

for some holomorphic function h . Then we have

$$\Omega^{n+l} = \det \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right) \mathbb{D} e^{-\sum_{i=1}^n \sigma_i x^i} |h|^2 (\sum dz^i \wedge d\bar{z}^i)^{n+l}.$$

Denote $\mathbb{F}_\Delta = \det(f_{ij})\mathbb{D}$. Choose $\{S_i, \bar{S}_i, S_\alpha, \bar{S}_\alpha, i \leq n, \alpha \in R_{M^+}\}$ as a local frame field. The Ricci tensor can be written as

$$(4.3) \quad \sum_{A,B} Ric\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial \bar{z}^B}\right) dz^A \wedge d\bar{z}^B = - \sum_{A,B} \left[\log \mathbb{F}_\Delta - \sum_{i=1}^n \sigma_i x^i \right]_{,A\bar{B}} \omega^A \wedge \bar{\omega}^B.$$

where “,” denotes the covariant derivatives of \mathcal{G}_f with respect to this frame field.

Since $\nabla_{\bar{S}_A} S_B$ is $(1,0)$ -vector field and $\mathcal{G}(S_\beta, \bar{S}_l) = 0$, by Koszul formula, we have

$$(4.4) \quad \begin{aligned} & \mathcal{G}(\nabla_{\bar{S}_\alpha} S_\beta, \frac{\partial}{\partial x^l}) = \mathcal{G}(\nabla_{\bar{S}_\alpha} S_\beta, \bar{S}_l) \\ &= \frac{1}{2} [-\bar{S}_l \mathcal{G}(\bar{S}_\alpha, S_\beta) - \mathcal{G}(\bar{S}_\alpha, [S_\beta, \bar{S}_l]) + \mathcal{G}(S_\beta, [\bar{S}_l, \bar{S}_\alpha]) + \mathcal{G}(\bar{S}_l, [\bar{S}_\alpha, S_\beta])] \\ &= \frac{1}{2} \delta_{\alpha\beta} \left[-\frac{1}{4} \frac{\partial D_\alpha}{\partial x^l} - \sum_j \frac{f_{jl}}{2} M_\alpha^j \right] = -\frac{1}{4} \frac{\partial D_\alpha}{\partial x^l} \delta_{\alpha\beta}. \end{aligned}$$

where we used $\frac{\partial D_\alpha}{\partial x^l} = \sum \frac{\partial D_\alpha}{\partial \xi_k} \frac{\partial \xi_k}{\partial x^l} = 2 \sum_k M_\alpha^k f_{kl}$. Similarly we have

$$(4.5) \quad \mathcal{G}(\nabla_{\bar{S}_j} S_k, \frac{\partial}{\partial x^l}) = \mathcal{G}(\nabla_{\bar{S}_j} S_\alpha, \frac{\partial}{\partial x^l}) = \mathcal{G}(\nabla_{\bar{S}_\alpha} S_j, \frac{\partial}{\partial x^l}) = 0.$$

For any function F depending only on (x^1, \dots, x^n) , we have

$$F_{,A\bar{B}} = S_A \bar{S}_B F - \mathcal{G}(\nabla_{S_A} \bar{S}_B, \frac{\partial}{\partial x^l}) f^{kl} \frac{\partial F}{\partial x^k}.$$

Then using (4.4) and (4.5) we have

$$(4.6) \quad F_{,j\bar{k}} = \frac{1}{4} \frac{\partial^2 F}{\partial x^j \partial x^k}, \quad F_{,\alpha\bar{j}} = 0, \quad F_{,j\bar{\alpha}} = 0, \quad F_{,\alpha\bar{\beta}} = \delta_{\alpha\beta} \frac{1}{4} \sum f^{kl} \frac{\partial D_\alpha}{\partial x^k} \frac{\partial F}{\partial x^l}.$$

By (4.3) and (4.6), the Ricci curvatures are given by

$$(4.7) \quad Ric(S_j, \bar{S}_k) = -\frac{1}{4} \frac{\partial^2 \log \mathbb{F}_\Delta}{\partial x^j \partial x^k}, \quad Ric(S_\alpha, \bar{S}_k) = Ric(S_j, \bar{S}_\alpha) = 0,$$

$$(4.8) \quad Ric(S_\alpha, \bar{S}_\beta) = \delta_{\alpha\beta} \left[-\frac{1}{4} \sum f^{kl} \frac{\partial D_\alpha}{\partial x^k} \frac{\partial \log \mathbb{F}_\Delta}{\partial x^l} + \frac{1}{4} \sum \frac{\partial D_\alpha}{\partial \xi_k} \sigma_k \right],$$

The scalar curvature can be written as

$$(4.9) \quad \mathbb{S} = - \sum_{i,j} f^{ij} \frac{\partial^2 \log \mathbb{F}_\Delta}{\partial x^i \partial x^j} - \sum_{k,l} f^{kl} \frac{\partial \log \mathbb{D}}{\partial x^k} \frac{\partial \log \mathbb{F}_\Delta}{\partial x^l} + h_G.$$

In terms of ξ and $u(\xi)$, \mathbb{S} can be written as

$$(4.10) \quad \mathbb{S} = -\frac{1}{\mathbb{D}} \sum_{i,j=1}^n \frac{\partial^2 \mathbb{D} u^{ij}}{\partial \xi_i \partial \xi_j} + h_G.$$

Here

$$\mathbb{D} = \prod_{\alpha \in R_{M^+}} D_\alpha, \quad h_G = \sum \sigma_i \frac{\partial \log \mathbb{D}}{\partial \xi_i} = \sum_{i,j} f^{ij} \sigma_i \frac{\partial \log \mathbb{D}}{\partial x^j},$$

where σ is the sum of the positive roots of R_M^+ , and

$$\sigma_i = -2 \sum_{\alpha \in R_M^+} \alpha(\sqrt{-1} \tilde{H}_i).$$

Put $A = \mathbb{S} - h_G$. We will consider the PDE

$$(4.11) \quad -\frac{1}{\mathbb{D}} \sum_{i,j=1}^n \frac{\partial^2 \mathbb{D} u^{ij}}{\partial \xi_i \partial \xi_j} = A.$$

The equation (4.11) was introduced by Donaldson [15] in the study of the scalar curvature of toric fibration, see also [21] and [19]. We call (4.11) a generalized Abreu Equation.

5. UNIFORM STABILITY

We introduce several classes of functions. Set

$$\mathcal{C} = \{u \in C(\bar{\Delta}) : u \text{ is convex on } \bar{\Delta} \text{ and smooth on } \Delta\},$$

$$\mathbf{S} = \{u \in C(\bar{\Delta}) : u \text{ is convex on } \bar{\Delta} \text{ and } u - v \text{ is smooth on } \bar{\Delta}\},$$

where v is given in (3.9). For a fixed point $p_o \in \Delta$, we consider

$$\mathcal{C}_{p_o} = \{u \in \mathcal{C} : u \geq u(p_o) = 0\},$$

$$\mathbf{S}_{p_o} = \{u \in \mathbf{S} : u \geq u(p_o) = 0\}.$$

We say functions in \mathcal{C}_{p_o} and \mathbf{S}_{p_o} are *normalized* at p_o .

We consider the generalized Abreu equation (4.11), where, $\mathbb{D} > 0$ and A are given smooth functions on $\bar{\Delta}$. Following [19] we consider the functional

$$(5.1) \quad \mathcal{F}_A(u) = - \int_{\Delta} \log \det(u_{ij}) \mathbb{D} d\mu + \mathcal{L}_A(u),$$

where

$$(5.2) \quad \mathcal{L}_A(u) = \int_{\partial\Delta} u \mathbb{D} d\sigma - \int_{\Delta} A u \mathbb{D} d\mu.$$

\mathcal{F}_A is called the Mabuchi functional and \mathcal{L}_A is closely related to the Futaki invariants. The Euler-Lagrangian equation for \mathcal{F}_A is (4.11). It is known that, if $u \in \mathbf{S}$ satisfies the equation (4.11), then u is an absolute minimizer for \mathcal{F}_A on \mathbf{S} .

Definition 5.1. Let $\mathbb{D} > 0$ and A be smooth functions on $\bar{\Delta}$. Then, (Δ, \mathbb{D}, A) is called *uniformly K-stable* if the functional \mathcal{L}_A vanishes on affine-linear functions and there exists a constant $\lambda > 0$ such that, for any $u \in \mathcal{C}_{p_o}$,

$$(5.3) \quad \mathcal{L}_A(u) \geq \lambda \int_{\partial\Delta} u \mathbb{D} d\sigma.$$

We also say that Δ is (\mathbb{D}, A, λ) -stable.

Using the same method in [10] we immediately get

Theorem 5.2. *If the equation (4.11) has a solution in \mathbf{S} , then (Δ, \mathbb{D}, A) is uniform K-stable.*

Donaldson [14] derived a L^∞ estimate for the Abreu's equation in $\Delta \subset \mathbb{R}^2$. His method can be applied directly to the generalized Abreu's Equation $\Delta \subset \mathbb{R}^2$ (see also [19]). We have

Theorem 5.3. *Let $\Delta \subset \mathbb{R}^2$ be a Delzant polytope, $\mathbb{D} > 0$ and A be two smooth functions defined on $\bar{\Delta}$. Let $u \in C^\infty(\Delta)$ satisfying (4.11). Suppose that Δ is (\mathbb{D}, A, λ) -stable. Then there is a constant $C_o > 0$, depending on λ , Δ , \mathbb{D} and $\|A\|_{C^0}$, such that $|\max_{\bar{\Delta}} u - \min_{\bar{\Delta}} u| \leq C_o$.*

6. DIFFERENTIAL INEQUALITIES

We first introduce some notations. Let p be a vertex of Δ , the edges meeting p are the form $p + tE_i$, $t \geq 0$, $E^i \in \mathbb{Z}^n$, $i = 1, \dots, n$. Consider the base $\{p; E^1, \dots, E^n\}$, let $\xi^p = (\xi_1^p, \dots, \xi_n^p)$ be the coordinates with respect to the base $\{p; E^i\}$. Suppose that

$$p = \sum_{i=1}^n c_i \nu^i, \quad \nu^i = \sum_{j=1}^n a_j^i E^j, \quad (a_j^i) \in SL(n, \mathbb{Z}),$$

we have the coordinate transformation

$$\xi_i^p = \sum_{j=1}^n a_i^j (\xi_j - c_i), \quad \xi^p(p) = 0, \quad \Delta \subset \{\xi^p | \xi_i^p > 0\}.$$

Set

$$x_p^i = \frac{\partial u}{\partial \xi_i^p}, \quad f_p(x_p^1, \dots, x_p^n) = \sum x_p^i \xi_i^p - u.$$

Then f_p can be naturally extend to a smooth function on a neighborhood U_p of $\tau_M^{-1}(p)$.
Let

$$w^i = x_p^i + \sqrt{-1}y_p^i, \quad z_p^i = e^{\frac{w^i}{2}}, \quad i = 1, \dots, n.$$

Then (z_p^1, \dots, z_p^n) is a local holomorphic coordinates of U_p . We have (see [8])

$$(6.1) \quad f_p = f - \sum_{i=1}^n c_i x^i.$$

Denote

$$(6.2) \quad \mathbb{F}_p = 4^{2n} \det \left(\frac{\partial^2 f_p}{\partial z_p^i \partial \bar{z}_p^j} \right) \mathbb{D} = \det \left(\frac{\partial^2 f_p}{\partial x_p^i \partial x_p^j} \right) e^{-x_p^1 - x_p^2 - \dots - x_p^n} \mathbb{D}.$$

$$(6.3) \quad \Psi_p := \|\nabla \log \mathbb{F}_p\|_f^2, \quad P = \exp(\kappa \mathbb{F}_p^a) \sqrt{\mathbb{F}_p} \Psi_p,$$

where a and κ are positive constants to be determined later.

More general, for each $(n-k)$ -dimensional face of Δ , one can associate it a complex coordinate chart of M :

$$\mathbb{U}_k \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}.$$

Let (w^1, \dots, w^n) , $w^i = x^i + \sqrt{-1}y^i$, be the log-affine coordinate of M . Set

$$z^i = e^{\frac{w^i}{2}}, \quad i \leq k.$$

Then $(z^1, \dots, z^k, w^{k+1}, \dots, w^n)$ is local holomorphic coordinates of \mathbb{U}_k . Denote

$$Z_{n-k} = \{p \in M | z^i = 0, i \leq k\}, \quad E = \{\xi | \xi_i = 0, i \leq k\},$$

where $\xi_i = \frac{\partial f}{\partial x^i}$. Let $f_E = f - \sum_{i=1}^k c_i x^i - d$, where c_i, d are constants such that $\tau_{f_E}(Z_{n-k}) = E$. Then f_E can be naturally extend to a smooth function in \mathbb{U}_k . Denote

$$(6.4) \quad \mathbb{F}_E = \det \left(\frac{\partial^2 f_E}{\partial x^i \partial x^j} \right) e^{-\sum_{i=1}^k x^i} \mathbb{D}.$$

$$(6.5) \quad \Psi_E := \|\nabla \log \mathbb{F}_E\|_f^2, \quad P_E = \exp(\kappa \mathbb{F}_E^a) \sqrt{\mathbb{F}_E} \Psi_E.$$

Set $V_E = \log \mathbb{F}_E$, $\mathbb{A}_E = -\square \log \mathbb{F}_E$.

Let Ω_g be the Guillemin metric on $G \times_K M$ with local potential function g . Denote by $\dot{R}_{i\bar{j}k\bar{l}}$ and $\dot{R}_{i\bar{j}}$ the curvature tensor and the Ricci curvature of Ω_g , respectively. Put

$$\dot{\mathcal{R}} := \sqrt{\sum g^{m\bar{n}} g^{k\bar{l}} g^{i\bar{j}} g^{s\bar{t}} \dot{R}_{m\bar{l}i\bar{t}} \dot{R}_{k\bar{n}s\bar{j}}}.$$

Let Ω_f be a Kähler metric on $G \times_K M$ in the same Kähler class as Ω_g with local potential function f . Then there is a global defined function $\phi \in C^\infty(G \times_K M)$ such that $\phi = f - g$. Denote $\square = \sum f^{A\bar{B}} \frac{\partial^2}{\partial z^A \partial \bar{z}^B}$ the Laplacian operator on $G \times_K M$ with respect to the metric Ω_f . Obviously $n - \square\phi = \sum f^{i\bar{j}} g_{i\bar{j}} := T$. Put

$$Q = e^{-N_1(\phi - \inf \phi + 1)} \sqrt{\mathbb{F}_p} T,$$

where $N_1 > 0$ is a constant.

In this section we establish some differential inequalities in $G(U_p)$ for $\log \mathbb{F}$, P and Q .

6.1. Subharmonic function $\log \mathbb{F}_p + Nf_p$. For any function F depending only on (x^1, \dots, x^n) we have

$$(6.6) \quad \square F = \sum f^{ij} F_{ij} + \sum f^{ij} \frac{\log \mathbb{D}}{\partial x^i} F_j = \sum f^{ij} F_{ij} + \sum \frac{\log \mathbb{D}}{\partial \xi_j} F_j.$$

where $F_i = \frac{\partial F}{\partial x^i}$, $F_{ij} = \frac{\partial^2 F}{\partial x^i \partial x^j}$.

Lemma 6.1. Choose $\{o, \nu^i, i = 1, \dots, n\}$ as a base of \mathbb{R}^n , let $\bar{\Delta} \subset \{(\xi_1, \dots, \xi_n) | \xi_1 > 0, \xi_2 > 0, \dots, \xi_n > 0\}$ be a Delzant polytope satisfying

$$(6.7) \quad \sum_{\alpha \in R_{M+}} \frac{\sum_{j=1}^n M_\alpha^j \text{diam}(\Delta)}{D_\alpha} < \frac{n}{4}.$$

Then there is a constant $N > 0$ depending only on n, \mathbb{D}, Δ and the position of Δ in \mathbb{R}^n such that for any vertex p of Δ

$$\square(\log \mathbb{F}_p + Nf_p) > 0.$$

Proof. By (6.1) we have

$$(6.8) \quad \begin{aligned} \square f_p &= \square(f - \sum c_i x^i) = n + \sum \frac{\partial \log \mathbb{D}}{\partial \xi_i} \left(\frac{\partial f}{\partial x^i} - c_i \right) \\ &\geq n - \sum_{\alpha \in R_{M+}} \frac{\sum_j M_\alpha^j \text{diam}(\Delta)}{\sum_j M_\alpha^j \xi_j} \geq \frac{n}{2}. \end{aligned}$$

As $x^i = \frac{\partial u}{\partial \xi_i} = \sum_j \frac{\partial u}{\partial \xi_j^p} \frac{\partial \xi_j^p}{\partial \xi_i} = \sum_j a_i^j x_p^j$, we have $\det \left(\frac{\partial^2 f}{\partial x_p^i \partial x_p^j} \right) = \det \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right)$. Denote by (b_i^j) the inverse of (a_i^j) . Then by (4.10) and (6.6)

$$(6.9) \quad \square \log \mathbb{F}_p = -\mathbb{S} + \sum_{k=1}^m \frac{\partial \log \mathbb{D}}{\partial \xi_k^p} (\sigma_k - \sum_j b_k^j) := -\mathbb{A}_p.$$

Denote \mathcal{P} be the set of all vertices of $\bar{\Delta}$. Let N be the constant such that

$$\max_{p \in \mathcal{P}} \max_{\xi \in \bar{\Delta}} \mathbb{A}_p < \frac{n}{2} N.$$

Then for any $p \in \mathcal{P}$, we have

$$\square(\log \mathbb{F}_p + N f_p) > 0.$$

■

6.2. Differential inequalities for P . We are going to calculate $\square P$ and derive a differential inequality. This inequality is similar to Lemma 5.1 in [7]. In this subsection we denote $\mathbb{F}_E, \mathbb{A}_E, \Psi_E, \dots$ by $\mathbb{F}, \mathbb{A}, \Psi, \dots$, etc. By (4.6), (4.7) and (4.8) we have

$$(6.10) \quad (\log \mathbb{F})_{,j\bar{k}} = -Ric(S_j, \bar{S}_k), \quad (\log \mathbb{F})_{,\alpha\bar{k}} = 0, \quad (\log \mathbb{F})_{,k\bar{\alpha}} = 0,$$

$$(6.11) \quad (\log \mathbb{F})_{,\alpha\bar{\beta}} = -Ric(S_\alpha, \bar{S}_\beta) + \frac{\delta_{\alpha\beta}}{4} h_\alpha,$$

where $h_\alpha = \sum_k \frac{\partial D_\alpha}{\partial \xi_k} \left(\sigma_k - \sum_j c_k^j \right)$ and c_k^j are constants depending only on E . Denote by $f_{,A\bar{B}}$ the components of the metric \mathcal{G} with respect to the frame $\{S_A, \bar{S}_B\}$, and by $(f^{A\bar{B}})$ the inverse of $(f_{,A\bar{B}})$. Put $W = \det(f_{,i\bar{j}})$, $V = \log \mathbb{F}$,

$$\|V_{,A\bar{B}}\|_f^2 = \sum f^{A\bar{B}} f^{C\bar{D}} V_{A\bar{D}} V_{B\bar{C}}, \quad \|V_{,AB}\|_f^2 = \sum f^{A\bar{B}} f^{C\bar{D}} V_{,AC} V_{,\bar{B}\bar{D}}.$$

Lemma 6.2.

$$\frac{\square P}{P} \geq \frac{\|V_{,A\bar{B}}\|_f^2}{2\Psi} + a^2 \kappa (1 - 2\kappa \mathbb{F}^a) \mathbb{F}^a \Psi - \frac{2|\langle \nabla \mathbb{A}, \nabla V \rangle|}{\Psi} - (a\kappa \mathbb{F}^a + \frac{1}{2}) \mathbb{A},$$

where \langle, \rangle denotes the inner product with respect to the metric Ω_f .

Proof. By definition,

$$\square \Psi = \sum f^{A\bar{B}} f^{C\bar{D}} (V_{,A} V_{,\bar{B}\bar{C}\bar{D}} + V_{,AC\bar{D}} V_{,\bar{B}} + V_{,AC} V_{,\bar{B}\bar{D}} + V_{,A\bar{D}} V_{,\bar{B}\bar{C}}).$$

Since V depends only on (x^1, \dots, x^n) , we have

$$(6.12) \quad V_{,\alpha} = V_{,\bar{\alpha}} = 0.$$

By the Ricci identities, (6.12) and $f^{j\bar{\alpha}} = 0$, we have

$$V_{,\bar{B}\bar{C}\bar{D}} = V_{,C\bar{D}\bar{B}}, \quad V_{,AC\bar{D}} = V_{,C\bar{D}A} + \sum f^{j\bar{h}} V_{,j} R_{C\bar{h}A\bar{D}}.$$

It follows that

$$(6.13) \quad \square \Psi = \|V_{,AB}\|^2 + \|V_{,A\bar{B}}\|^2 + \sum f^{i\bar{j}} f^{k\bar{l}} (-V_{i\bar{l}} V_{,k} V_{,\bar{j}}) - 2\text{Re}(\sum f^{i\bar{j}} V_{,i} \mathbb{A}_{,\bar{j}}),$$

where we use the facts $R_{i\bar{j}} = -V_{k\bar{l}}$ and $\square V = -\mathbb{A}$. Denote $\Pi = a\kappa\mathbb{F}^a + \frac{1}{2}$. Then

$$(6.14) \quad \begin{aligned} P_{,i} &= P \left(\frac{\Psi_{,i}}{\Psi} + \Pi V_{,i} \right) =: P \Lambda_i, \\ \square P &= P \left[\sum f^{i\bar{j}} \Lambda_i \Lambda_{\bar{j}} + \frac{\square \Psi}{\Psi} - \frac{\|\nabla \Psi\|_f^2}{\Psi^2} + \Pi \square V + a^2 \kappa \mathbb{F}^a \Psi \right]. \end{aligned}$$

For any point $q \in G(\mathbf{U}_k)$, we choose an affine transformation of the frame fields $\{S_A, \bar{S}_B\}$ such that, at q ,

$$f_{,i\bar{j}} = c\delta_{ij}, \quad 1 \leq i, j \leq n, \quad V_1 = V_{\bar{1}}, \quad V_i = V_{\bar{i}} = 0, \quad \forall 1 \leq i \leq n,$$

where $c = [W(q)]^{\frac{1}{n}}$. Then by the same arguments of Lemma 5.1 in [7] we can prove the lemma. ■

6.3. Differential inequality for Q . The following differential inequality of $n - \square\phi$ has been proved in [7] (see Section §5, Inequality-II)

Lemma 6.3.

$$(6.15) \quad \square \log(n - \square\phi) \geq -\|Ric\|_f - \dot{\mathcal{R}}(n - \square\phi).$$

In the following we restrict ourself to $n = 2$.

Lemma 6.4. *Suppose that*

$$(6.16) \quad \|\mathbb{A}_p\|_{C^1(\tau_f(U_p))} \leq \mathbf{N}_2, \quad \max_{\bar{U}_p} \mathbb{F} \leq \mathbf{N}_2, \quad \max_{\bar{U}_p} |\phi| + |z| \leq \mathbf{N}_2$$

for some constant $\mathbf{N}_2 > 0$. Then we may choose

$$(6.17) \quad \mathbf{N}_1 = 100, a = \frac{1}{3}, \kappa = [4\mathbf{N}_2^{\frac{1}{3}}]^{-1}$$

such that

$$(6.18) \quad \square(P + Q + C_1 f_p) \geq C_2(P + Q)^2 > 0$$

for some positive constants C_1 and C_2 that depend only on \mathbf{N}_2 , the structure constants of \mathfrak{g} , \mathbb{D} , Δ and the position of Δ in \mathbb{R}^2 .

Proof. Applying Lemma 6.2 and the choice of $a = \frac{1}{3}$ and κ , in particular, $\kappa\mathbb{F}^a \leq 1/4$, we have

$$(6.19) \quad \frac{\Psi \square P}{P} \geq \left(\frac{1}{2} \|V_{,A\bar{B}}\|_f^2 + \frac{1}{18} \kappa \mathbb{F}^{\frac{1}{3}} \Psi^2 \right) - (2|\langle \nabla \mathbb{A}, \nabla V \rangle| + \Psi |\mathbb{A}|).$$

Treatment for $\langle \nabla \mathbb{A}, \nabla V \rangle$: using log-affine coordinates we have

$$|\langle \nabla \mathbb{A}, \nabla V \rangle| = \left| \sum f^{ij} \frac{\partial \mathbb{A}}{\partial x^i} \frac{\partial V}{\partial x^j} \right| = \left| \sum \frac{\partial K}{\partial \xi_k} \frac{\partial V}{\partial x^k} \right| \leq \mathbf{N}_2 \sum_j \left| \frac{\partial V}{\partial x^j} \right|.$$

If we use the complex coordinates z_i , we have

$$\left| \frac{\partial V}{\partial x^j} \right| = \left| z_j \frac{\partial V}{\partial z_j} \right|.$$

Since $|z|$ is bounded, in this coordinates we have

$$C^{-1} \leq g_{i\bar{j}} \leq C, \quad \sum f^{i\bar{i}} \leq C \sum f^{i\bar{j}} g_{i\bar{j}} \leq CT.$$

Then we conclude that

$$|\langle \nabla \mathbb{A}, \nabla V \rangle| \leq C \sqrt{2\mathbb{F}T\Psi}.$$

We explain the last step: suppose that $0 < \nu_1 \leq \nu_2$ are the eigenvalues of $(f_{i\bar{j}})$, then

$$\Psi = \sum f^{i\bar{j}} V_i V_{\bar{j}} \geq \nu_2^{-1} (|V_1|^2 + |V_2|^2) \geq (\mathbb{F}T)^{-1} (|V_1|^2 + |V_2|^2).$$

Note that

$$(e\mathbb{F})^{-\frac{1}{2}} \leq \Psi/P = (\exp(\kappa\mathbb{F}^\alpha)\mathbb{F}^{\frac{1}{2}})^{-1} \leq \mathbb{F}^{-\frac{1}{2}},$$

(6.19) is then transformed to be

$$(6.20) \quad \square P \geq \mathbb{F}^{\frac{1}{2}} \left(\frac{1}{2} \|V_{,A\bar{B}}\|_f^2 + \frac{1}{18} \kappa \mathbb{F}^{\frac{1}{3}} \Psi^2 \right) - C' \mathbb{F}^{\frac{1}{2}} \left(\sqrt{\mathbb{F}T\Psi} + \Psi |\mathbb{A}| \right).$$

Applying the Young inequality and the Schwartz inequality to terms in (6.20), we have that

$$(6.21) \quad \square P \geq \frac{1}{2} \mathbb{F}^{\frac{1}{2}} \|V_{,A\bar{B}}\|_f^2 + C_1 \mathbb{F}^{-\frac{1}{6}} P^2 - \epsilon QT - C_2(\epsilon).$$

By (6.10), (6.11) and Cauchy inequalities, for any $\delta \in (0, 1)$, we obtain that

$$\|Ric\|_{\mathcal{G}_f}^2 = \sum_{i,j} \|V_{i\bar{j}}\|^2 + \sum_{\alpha \in R_{M^+}} \|V_{,\alpha\bar{\alpha}} + h_\alpha\|^2 \leq (1 + \delta) \|V_{,A\bar{B}}\|^2 + C_{\delta, R_{M^+}},$$

where $C_{\delta, R_{M^+}} > 0$ is a constant depending only on $1/\delta$ and $\sum_{\alpha \in R_{M^+}} h_\alpha^2$. By a direct calculation and the formula (6.15) we have

$$\begin{aligned} \square Q &\geq Q \left(N_1 T + \frac{1}{2} \square V + \square \log T \right) \\ &\geq Q \left((N_1 - \dot{\mathcal{R}}) T - \frac{1}{2} \mathbb{A} - (1 + \delta) \|V_{,A\bar{B}}\|_f - C_{\delta, R_{M^+}} \right), \end{aligned}$$

Choosing δ small, using the explicit value N_1 and the bounds of \mathbb{F} and \mathbb{A} , applying the Schwartz inequality properly, we can get

$$(6.22) \quad \square Q \geq -\frac{1}{4}\mathbb{F}^{\frac{1}{2}}\|V_{A\bar{B}}\|_f^2 + \frac{N_1}{3}QT - C_3(N_1, N_2, \delta, R_{M^+}).$$

Combining (6.21) and (6.22), and choosing $\epsilon = \frac{1}{100}$, we have

$$\square(Q + P) \geq C_1\mathbb{F}^{-\frac{1}{6}}P^2 + \frac{N_1}{4}QT - C_4.$$

Note that

$$T = e^{-N_1\phi}\mathbb{F}^{-\frac{1}{2}}Q \geq e^{-N_1\phi}N_2^{-\frac{1}{3}}\mathbb{F}^{-\frac{1}{6}}Q \geq C_5\mathbb{F}^{-\frac{1}{6}}Q,$$

we get

$$\square(Q + P) \geq C_6\mathbb{F}^{-\frac{1}{6}}(Q + P)^2 - C_7$$

where C_6, C_7 are constants depending only on C_1, N_1 and N_2 . Our lemma follows from $\square f_p \geq n/2$. and $|\mathbb{F}| \leq N_2$. ■

6.4. Interior estimate of Ψ . We use Lemma 6.2 to derive the interior estimate of Ψ in a geodesic ball in U_k . In this subsection we denote $\mathbb{F}_E, \mathbb{A}_E, \Psi_E, \dots$ by $\mathbb{F}, \mathbb{A}, \Psi, \dots$, etc.

Lemma 6.5. *Let $B_a(o) \subset U_k$ be a closed geodesic ball of radius a centered at o with respect to the metric \mathcal{G}_f . Set $\mathbb{F}_\diamond := \max_{B_a(o)} \mathbb{F}$. Suppose that*

$$(6.23) \quad \min_{B_a(o)} |\mathbb{A}| \neq 0, \text{ and, } \mathbb{F}^{\frac{1}{2}}(\mathcal{K} + \mathbb{K} + \|\nabla \log |\mathbb{A}|\|_f^2 + \Psi) \leq 4,$$

in $B_a(o)$. Then the following estimate holds in $B_{a/2}(o)$

$$(6.24) \quad \mathbb{F}^{\frac{1}{2}}\Psi \leq C_6 \left[\mathbb{F}_\diamond^{\frac{1}{2}} \max_{B_a(o)} |\mathbb{A}| + \mathbb{F}_\diamond^{\frac{1}{3}} \max_{B_a(o)} |\mathbb{A}|^{\frac{2}{3}} + a^{-1}\mathbb{F}_\diamond^{\frac{1}{4}} + a^{-2}\mathbb{F}_\diamond^{\frac{1}{2}} \right],$$

where C_6 is a constant depending only on n .

Proof. Consider the function

$$F := (a^2 - r^2)^2 P$$

defined in $B_a(o)$, where r denotes the geodesic distance from o to z with respect to the metric \mathcal{G}_f . F attains its supremum at some interior point q^* . Choose $\kappa = \frac{1}{4\mathbb{F}_\diamond^{\frac{1}{4}}}$, $\alpha = \frac{1}{4}$, As in [7], using Lemma 6.2 and a direct calculation we have, at q^* ,

$$(6.25) \quad \epsilon_o \left(\frac{\mathbb{F}}{\mathbb{F}_\diamond} \right)^{\frac{1}{4}} \Psi - \frac{2\|\nabla \mathbb{A}\|_f}{\sqrt{\Psi}} - \frac{9}{16}|\mathbb{A}| - \frac{24a^2}{(a^2 - r^2)^2} - \frac{4(1 + r\square r)}{a^2 - r^2} \leq 0,$$

where $\epsilon_o = \frac{1}{128}$. Denote by Γ the geodesic from o to q^* . To estimate $r\square r$, we consider two cases:

Case 1. Along Γ the following estimate holds

$$\mathbb{F} \geq \frac{1}{100} \mathbb{F}(q^*).$$

Case 2. There is a point $q \in \Gamma$ such that $\mathbb{F}(q) < \frac{1}{100} \mathbb{F}(q^*)$. Then there is a point $q_1 \in \Gamma$ such that

$$\mathbb{F}(q_1) = \frac{1}{100} \mathbb{F}(q^*), \quad \mathbb{F}(q) \geq \frac{1}{100} \mathbb{F}(q^*), \quad \forall q \in [q_1, q^*].$$

Then by the same argument of [7] for both cases we have estimates (6.24) in $B_{a/2}(o)$. ■

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